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The BRS identities for the general axial gauge in quantum gravity

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Abstract. We derive the BRS identities for quantum gravity in the axial gauge and use them to explain why the one-loop counterterm is *not* generally covariant.

1. Introduction

Recent investigations have clarified the invariance properties of the Yang–Mills theory in a general axial gauge. To review the situation briefly we consider a Lagrangian of the form‡

$$\mathscr{L}_{\mathrm{YM}} = -\tfrac{1}{4}(F^a_{\mu\nu})^2 + \frac{1}{2\alpha n^2} n_\mu A^a_\mu \partial^2 n_\nu A^a_\nu. \tag{1.1}$$

The case $\alpha = 0$ corresponds to the axial gauge whereas the $\alpha = 1$ case is known as the planar gauge (Dokshitzer *et al* 1980). Such a Lagrangian gives rise to ghosts, but within the context of dimensional regularisation it can be shown that closed loops of ghosts are zero and hence do not contribute to the S matrix. It might be thought that this implies that Green functions satisfy the classical Ward identities and hence that the counterterm is of the form $(F^a_{\mu\nu})^2$, even for $\alpha \neq 0$. In fact two somewhat different approaches have been used to show that this is not correct. Capper and Leibbrandt (1981a, b) show that, due to the nonlinearity of the gauge transformation, the Ward identities are complicated by the occurrence of the so-called 'pincer diagrams', even though ghosts are absent. Andraši and Taylor (1981) and Fadin and Milstein (1981) on the other hand, show that although the ghosts decouple from the S matrix they do not decouple from the BRS identities (Becchi *et al* 1975). Even though entirely different diagrams are involved in the evaluation of the counterterms, both approaches are in complete agreement for general α. It is well known, of course, that in the axial gauge ($\alpha = 0$) the counterterm is of the form $(F^a_{\mu\nu})^2$. This is because the ghosts decouple from both the S matrix and the BRS identities on the imposition of the $n_\mu A^a_\mu = 0$ gauge condition. Hence the argument that closed loops of ghosts are zero due to dimensional regularisation is superfluous in this particular case.

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‡ We use a $+---$ metric and dimensionally regularise all integrals in a 2ω-dimensional space–time. $\delta_{\mu\nu}$ denotes our flat-space metric 'tensor'.

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The axial gauge can also be used in quantum gravity by employing a Lagrangian of the form

$$\mathscr{L}_{\mathrm{E}}=2\sqrt{-g}K^{-2}R+\frac{1}{2\alpha}(n_\mu\phi_{\mu\nu})^2 \tag{1.2}$$

where R is the Ricci scalar and the graviton field, $\phi_{\mu\nu}$, is defined in terms of the metric tensor by

$$g_{\mu\nu}=\delta_{\mu\nu}+K\phi_{\mu\nu}. \tag{1.3}$$

Again the axial gauge is given by the limit $\alpha\to 0$. Unfortunately, examination of the Ward identities (Capper and Leibbrandt 1981c) shows that, even in this limit, the counterterms in quantum gravity are not generally covariant objects (such as $\sqrt{-g}R^2$, etc). Not only do the counterterms depend on the parameter n_μ (as also shown by Delbourgo 1981), but an explicit calculation shows that the graviton self-energy is non-transverse. These difficulties are due to the non-vanishing of the contribution to the Ward identities from the pincer diagram (figure 1) in the $\alpha\to 0$ limit. This contrasts with the Yang–Mills case where the pincer diagram *decouples* in the $\alpha\to 0$ limit. In the present paper we want to complete the study of the axial gauge in quantum gravity by carrying out an analysis analogous to that done by Andraši and Taylor (1981) for the Yang–Mills Lagrangian; that is we wish to examine the BRS rather than Ward identities. Again it turns out that completely different diagrams need to be calculated but the results are entirely compatible.

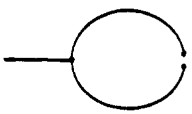


Figure 1. The pincer diagram which occurs in the Ward identities of Capper and Leibbrandt (1981c). The full curves and line represent gravitons.

2. Derivation of the BRS identities

Using the parametrisation of equation (1.3), the Einstein action is invariant under the gauge transformation

$$\delta\phi_{\mu\nu}=A_{\mu\nu\lambda}\xi_\lambda \tag{2.1}$$

where

$$A_{\mu\nu\lambda}=K^{-1}(\delta_{\nu\lambda}\partial_\mu+\delta_{\mu\lambda}\partial_\nu)+(\phi_{\lambda\nu}\partial_\mu+\phi_{\lambda\mu}\partial_\nu+\partial_\lambda\phi_{\mu\nu}) \tag{2.2}$$

and $\xi_\lambda(x)$ is an arbitrary gauge parameter. The ghost Lagrangian $\mathscr{L}_{\mathrm{G}}$ is given by

$$\mathscr{L}_{\mathrm{G}}=\bar{C}_\nu n_\mu A_{\mu\nu\lambda}C_\lambda \tag{2.3}$$

C_λ and $\bar{C}_\nu$ being anticommuting complex vector ghost fields. As can be seen, the ghosts do not decouple in $\mathscr{L}_{\mathrm{G}}$ even for $n_\mu\phi_{\mu\nu}=0$. This is in sharp contrast to the analogous Yang–Mills case. For non-zero α, the complete Lagrangian is given by

$$\mathscr{L}=\mathscr{L}_{\mathrm{E}}+\mathscr{L}_{\mathrm{G}} \tag{2.4}$$

and $\mathscr{L}$ is invariant under the BRS transformations

$$\delta\phi_{\mu\nu} = A_{\mu\nu\rho}C_\rho\Lambda \tag{2.5}$$

$$\delta\bar{C}_\beta = \alpha^{-1}\Lambda n_\mu \phi_{\mu\beta} \tag{2.6}$$

$$\delta C_\beta = C_\rho \Lambda \partial_\rho C_\beta \tag{2.7}$$

where Λ is a constant anticommuting parameter. We can introduce sources via

$$\mathscr{L}_J = J_{\mu\nu}\phi_{\mu\nu} + \bar{J}_\mu C_\mu + \bar{C}_\mu J_\mu + I_{\mu\nu}A_{\mu\nu\rho}C_\rho + I_\mu C_\rho \partial_\rho C_\mu \tag{2.8}$$

and define the generating functional for Green functions, Z, by

$$Z[J_{\mu\nu}, \bar{J}_\mu, J_\mu, I_{\mu\nu}, I_\mu] = \int \mathrm{d}[\phi_{\alpha\beta}]\,\mathrm{d}[\bar{C}_\mu]\,\mathrm{d}[C_\nu] \exp \mathrm{i} \int \mathrm{d}x\ (\mathscr{L} + \mathscr{L}_J). \tag{2.9}$$

As usual we define $W[J_{\mu\nu}, \bar{J}_\mu, J_\mu, I_{\mu\nu}, I_\mu]$ by

$$Z = \exp \mathrm{i} W \tag{2.10}$$

and Γ, the generating functional for proper vertices, via

$$W[J_{\mu\nu}, \bar{J}_\mu, J_\mu, I_{\mu\nu}, I_\mu] = \int \mathrm{d}x\ (J_{\mu\nu}\phi_{\mu\nu} + \bar{J}_\mu C_\mu + \bar{C}_\mu J_\mu) + \Gamma[\phi_{\mu\nu}, \bar{C}_\mu, C_\mu; I_{\mu\nu}, I_\mu]. \tag{2.11}$$

Invariance under the transformations of equations (2.5)–(2.7) then leads to the identity

$$\int \mathrm{d}x \left(-\frac{\delta\Gamma}{\delta\phi_{\mu\nu}} \frac{\delta\Gamma}{\delta I_{\mu\nu}} + \frac{\delta\Gamma}{\delta C_\mu} \frac{\delta\Gamma}{\delta I_\mu} + \frac{1}{\alpha} n_\beta \phi_{\beta\mu} \frac{\delta\Gamma}{\delta\bar{C}_\mu} \right) = 0. \tag{2.12}$$

Equation (2.12) contains all the BRS identities. Operating with $\delta^2/\delta\phi_{\rho\sigma}(x)\delta C_\lambda(y)$ we obtain†

$$\int \mathrm{d}z \left(-\frac{\delta^2\Gamma}{\delta\phi_{\rho\sigma}(x)\delta\phi_{\mu\nu}(z)} \frac{\delta^2\Gamma}{\delta I_{\mu\nu}(z)\delta C_\lambda(y)} + \frac{1}{\alpha} n_{(\rho}\delta_{\sigma)\mu}\delta(x-z) \frac{\delta^2\Gamma}{\delta C_\lambda(y)\delta\bar{C}_\mu(z)} \right) = 0. \tag{2.13}$$

In the tree approximation, equation (2.13) can be represented diagrammatically as shown in figure 2 and can be verified directly by using the Feynman rules given in table 1. The verification of equation (2.13) to one loop is carried out in the next section.

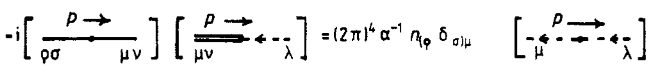


Figure 2. The BRS identity in the tree approximation. The broken lines represent ghosts and the double lines the $I_{\mu\nu}$ current. The arrows on the broken lines represent directed ghosts and do not correspond to the momenta directions.

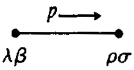
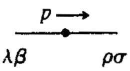
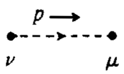
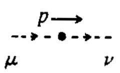
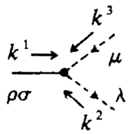
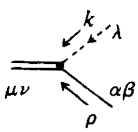
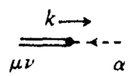
3. The verification of the BRS identity in the one-loop approximation

To one loop, equation (2.13) may be portrayed diagrammatically as shown in figure 3 and to verify the equation to this order we need to evaluate the diagrams defined in figures 4 and 5. The pole part of the graviton self-energy, $\bar{\Pi}_{\rho\sigma,\mu\nu}$, has already been

† We use the standard notation for symmetrised indices

$$A_{(\alpha}B_{\beta)} = \tfrac{1}{2}(A_\alpha B_\beta + A_\beta B_\alpha).$$

Table 1. Feynman rules for the propagators and vertices.

Functional representation	Diagrammatic representation	Momentum space expression
		Graviton propagator
$\left(\frac{-i\delta^2\Gamma}{\delta\phi_{\lambda\beta}\delta\phi_{\rho\sigma}}\right)^{-1}$		$G_{\lambda\beta,\rho\sigma} = \frac{-1}{2i(2\pi)^4 p^2} \times \left\{2I^1 - I^2 + 2\alpha\left(\frac{p^2}{p\cdot n}\right)^2\left[T^6+\frac{p^2n^2}{(p\cdot n)^2}T^9-4T^{10}\right]\right\}_{\lambda\beta,\rho\sigma}$
$\frac{i\delta^2\Gamma}{\delta\phi_{\lambda\beta}\delta\phi_{\rho\sigma}}$		$-G^{-1}_{\lambda\beta,\rho\sigma} = \frac{i(2\pi)^4p^2}{2}\left\{2T^1-2T^2+2T^3-T^6+\frac{n^2}{2\alpha p^2}T^8\right\}_{\lambda\beta,\rho\sigma}$
		Ghost propagator
$\left(\frac{-i\delta^2\Gamma}{\delta C_\nu\delta C_\mu}\right)^{-1}$		$G_{\mu\nu} = \frac{-1}{(2\pi)^4 p\cdot n}\left(\delta_{\mu\nu}-\frac{p_\mu n_\nu}{2p\cdot n}\right)$
$\frac{i\delta^2\Gamma}{\delta\bar{C}_\nu\delta C_\mu}$		$-G^{-1}_{\mu\nu} = -(2\pi)^4(p\cdot n\delta_{\mu\nu}+n_\mu p_\nu)$
		Ghost vertex
$\frac{i\delta^3\Gamma}{\delta\phi_{\rho\sigma}\delta C_\mu\delta\bar{C}_\lambda}$		$V_{\rho\sigma,\lambda\mu} = -(2\pi)^4\{\delta_{\lambda(\rho}\delta_{\sigma)\mu}n\cdot k^3+n_{(\rho}\delta_{\sigma)\mu}k^3_\lambda+n_{(\rho}\delta_{\sigma)\lambda}k'_\mu\}$
		$I_{\mu\nu}-C_\lambda-\phi_{\alpha\beta}$ vertex
$\frac{i\delta^3\Gamma}{\delta I_{\mu\nu}\delta C_\lambda\delta\phi_{\alpha\beta}}$		$\hat{V}_{\mu\nu,\lambda,\alpha\beta} = -(2\pi)^4\{2\delta_{\lambda(\alpha}\delta_{\beta)\nu}k_\mu+\delta_{\mu(\alpha}\delta_{\beta)\nu}p_\lambda\}$
		$I_{\mu\nu}-C_\alpha$ vertex
$\frac{i\delta^2\Gamma}{\delta I_{\mu\nu}\delta C_\alpha}$		$\bar{V}_{\mu\nu,\alpha} = (2\pi)^4(\delta_{\alpha\mu}k_\nu+\delta_{\alpha\nu}k_\mu)$

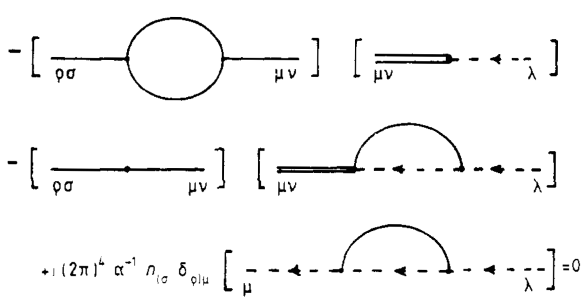


Figure 3. The BRS identity in the one-loop approximation.

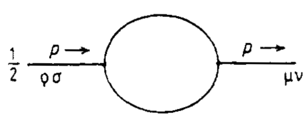


Figure 4. The one-loop gravition self-energy $\Pi_{\rho\sigma,\mu\nu}(p)$.

Figure 5. The diagram defining $W_{\mu\nu,\lambda}(p)$ in the one-loop approximation.

worked out by Capper and Leibbrandt (1981c) and the evaluation of $W_{\mu\nu,\lambda}$, which corresponds to figure 5, is carried out in a similar manner. As before, it is easier to contract $W_{\mu\nu,\lambda}$ with momenta or a Kronecker delta before evaluation. We thus obtain (for the pole parts)†

$$\bar{W}_1 = p_\mu \bar{W}_{\alpha\alpha,\mu} = \frac{8}{3\mathrm{i}}(p^2)^2 y(y-1)K^2\bar{I} \tag{3.1}$$

$$\bar{W}_2 = p_\alpha p_\beta p_\mu \bar{W}_{\alpha\beta,\mu} = -\frac{8}{3\mathrm{i}}(p^2)^3 y(y-1)^2 K^2\bar{I} \tag{3.2}$$

$$\bar{W}_9 = p_\alpha \bar{W}_{\alpha\beta,\beta} = \frac{16}{3\mathrm{i}}(p^2)^2 y(y-1)K^2\bar{I} \tag{3.3}$$

where

$$y = (p\cdot n)^2/p^2n^2 \tag{3.4}$$

and

$$\bar{I} = \text{divergent part of} \int \frac{\mathrm{d}^{2\omega}q}{q^2(p-q)^2} = \frac{-\mathrm{i}\pi^2}{(\omega-2)}. \tag{3.5}$$

The other seven independent scalars that can be formed out of $\bar{W}_{\mu\nu,\lambda}$ are all zero.

The equation shown diagrammatically in figure 3 thus reduces to the two equations

$$\bar{\Pi}_3 = \bar{\Pi}_{\rho\rho,\lambda\mu} p_\lambda p_\mu = -\mathrm{i}(p^2\bar{W}_1 - \bar{W}_2) \tag{3.6}$$

and

$$\bar{\Pi}_7 = \bar{\Pi}_{\rho\sigma,\rho\mu} n_\sigma p_\mu = \tfrac{1}{2}\mathrm{i}p\cdot n(\bar{W}_1 - \bar{W}_9). \tag{3.7}$$

Use of the results for $\bar{\Pi}_3$ and $\bar{\Pi}_7$ given by Capper and Leibbrandt (1981c) together with those given above for $\bar{W}_1$, $\bar{W}_2$ and $\bar{W}_9$ verifies equations (3.6) and (3.7). We note that all the amplitudes $\bar{\Pi}_3$, $\bar{\Pi}_7$, $\bar{W}_1$, $\bar{W}_2$ and $\bar{W}_9$ are independent of α, in spite of the calculations being carried out for the general gauge breaking term in equation (1.2). Furthermore, it is precisely the non-vanishing of $\bar{\Pi}_3$ and $\bar{\Pi}_7$ which leads to the non-transversality of the pole part of the graviton self-energy in the axial gauge, and thus the inadmissibility of generally covariant counterterms in this gauge.

4. The ghost equation of motion

It is also straightforward to derive the ghost equation of motion for the general axial gauge defined by equation (1.2) and it may be written as

$$n_\mu \frac{\delta\Gamma}{\delta I_{\mu\nu}} = \frac{\delta\Gamma}{\delta\bar{C}_\nu} \tag{4.1}$$

† The W_i given in equations (3.1)–(3.3) were calculated by computer using exactly the same set of integrals as in Capper and Leibbrandt (1981c). The bar superscript on the W_i and Π_i indicates that only the pole parts are considered.

which leads to the identity

$$n_\mu \frac{\delta^2\Gamma}{\delta C_\alpha \delta I_{\mu\nu}} = \frac{\delta^2\Gamma}{\delta C_\alpha \delta \bar{C}_\nu}. \tag{4.2}$$

The tree graph approximation to equation (4.2) is shown in figure 6 and may readily be verified by using the Feynman rules of table 1. The one-loop approximation is shown in figure 7. Explicit calculation shows that all contributions to figure 7 are in fact zero.

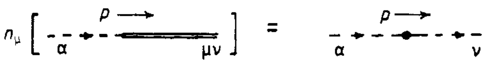


Figure 6. The ghost equation of motion in the tree approximation.

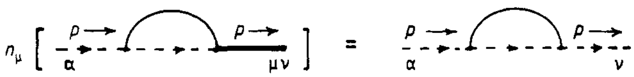


Figure 7. The ghost equation of motion in the one-loop approximation.

5. Conclusion

We have derived the BRS identities for the general axial gauge in quantum gravity and verified one set in the one-loop approximation. Unlike in a covariant gauge (Delbourgo and Ramon-Medrano 1976) a new diagram (i.e. figure 5) has to be calculated in order to verify the BRS rather than Ward identities.

The failure of the axial gauge to give rise to generally covariant counterterms in quantum gravity is, to say the least, highly inconvenient and it is worthwhile searching for some variation of the axial gauge that may overcome this difficulty. Studying the BRS rather than Ward identities is a useful way of tackling the problem since the first requirement is that the ghost fields decouple from the BRS identities. One possibility is to retain the form of the gauge breaking term given in equation (1.2) but to use a more general parametrisation of the graviton field by defining

$$\bar{\phi}_{\mu\nu} = [\lambda_1\phi_{\mu\nu} + \lambda_2\delta_{\mu\nu}\phi_{\alpha\alpha}] + K[\lambda_3\phi_{\mu\alpha}\phi_{\alpha\nu} + \lambda_4\phi_{\mu\nu}\phi_{\alpha\alpha} + \lambda_5\delta_{\mu\nu}\phi_{\alpha\alpha}\phi_{\beta\beta}].$$

Unfortunately a detailed examination shows that there are no values of the parameters λ_i such that the ghosts decouple. This confirms the conclusions of Capper and Leibbrandt (1981d).

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